The PML_2 Language

Integrated Program Verification in ML

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Semantics and Implementation of an Extension of ML for Proving Programs

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A Programming Language, with Program Proving Features

An ML-like programming language with:

- records, variants (constructors), inductive types,
- polymorphism, general recursion,
- a call-by-value evaluation strategy,
- effects (control operators),
- a light, Curry-style syntax and subtyping.

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For proving program, the type system is enriched with:

- programs as individuals (higher-order layer),
- an equality type $t \equiv u$ (observational equivalence),
- a dependent function type (typed quantification).
- Termination checking is required for proofs.

Example of Program and Proof

```
type rec nat = [Zero ; S of nat]

val rec add : nat ⇒ nat ⇒ nat =
  fun n m { case n { Zero → m | S[k] → S[add k m] } }
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val rec add_n_Zero : ∀n∈nat, add n Zero ≡ n =
  fun n {
    case n {
      Zero → {}
      S[p] → add_n_Zero p
    }
  }
```

Part I

Specific Type Constructors

Examples of (equational) program properties:

- `add (add m n) k` $\equiv$ `add m (add n k)` (associativity of `add`)
- `rev (rev l)` $\equiv$ `l` (`rev` is an involution)
- `map g (map f l)` $\equiv$ `map (fun x {g (f x)}) l` (`map` and composition)
- `sort (sort l)` $\equiv$ `sort l` (`sort` is idempotent)

Properties as Program Equivalences

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- `sort (sort l)` ≡ `sort l` (sort is idempotent)

Specification of a sorting function using predicates:

- `sorted (sort l)` ≡ `true` (sort produces a sorted list)
- `permutation (sort l) l` ≡ `true` (sort yields a permutation)

Equality Types and Equivalence

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Remark: cannot be complete since equivalence is undecidable.

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First-Order Quantification is not Enough

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We need a form of typed quantification!

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$$\frac{\Gamma, x : A ; \Xi \vdash t : B}{\Gamma ; \Xi \vdash \lambda x.t : \forall x {\in} A.B}$$

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```
val rec add_n_Sm : ∀n m∈nat, add n S[m] ≡ S[add n m] =
  fun n m {
    case n { Zero → {} | S[k] → add_n_Sm k m }
  }

val rec add_comm : ∀n m∈nat, add n m ≡ add m n =
  fun n m {
    case n {
      Zero → add_n_Zero m
      S[k] → add_n_Sm m k; add_comm k m
    }
  }
```

Part II

Formalisation of the System and Semantics

We build a model to prove that the language has the expected properties.

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To construct the model, we need to:

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3) define adequate typing rules,
4) deduce *termination*, *type safety* and *consistency*.

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4) deduce *termination*, *type safety* and *consistency*.

Advantage: it is a very flexible approach.

Values (Λ_ι)	$v, w ::= x \mid \lambda x.t \mid \{(l_i = v_i)_{i \in I}\} \mid C_k[v]$	
Terms (Λ)	$t, u ::= v \mid t\ u \mid v.l_k \mid [v \mid (C_i[x_i] \rightarrow t_i)_{i \in I}] \mid \mu\alpha.t \mid [\pi]t$	
Stacks (Π)	$\pi, \xi ::= \alpha \mid \varepsilon \mid v \,.\, \pi \mid [t]\pi$	(evaluation context)
Processes	$p, q ::= t * \pi$	

Call-by-value Reduction Relation

$$
\begin{aligned}
t\,u * \pi &\succ u * [t]\pi \\
v * [t]\pi &\succ t * v \,.\, \pi \\
\lambda x.t * v \,.\, \pi &\succ t[x := v] * \pi \\
\{(l_i = v_i)_{i \in I}\}.l_k * \pi &\succ v_k * \pi && (k \in I) \\
[C_k[v] \mid (C_i[x_i] \to t_i)_{i \in I}] * \pi &\succ t_k[x_k := v] * \pi && (k \in I) \\
\mu\alpha.t * \pi &\succ t[\alpha := \pi] * \pi \\
[\pi]t * \xi &\succ t * \pi
\end{aligned}
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$$[\![\{l_1 : A_1 ; l_2 : A_2\}]\!] = \Big\{\{l_1 = v_1 ; l_2 = v_2\} \mid v_1 \in [\![A_1]\!] \;\wedge\; v_2 \in [\![A_2]\!]\Big\}$$

$$[\![[C_1 : A_1 \mid C_2 : A_2]]\!] = \Big\{C_i[v] \mid i \in \{1, 2\} \;\wedge\; v \in [\![A_i]\!]\Big\}$$

$$[\![\forall X.A]\!] = \bigcap_{\Phi \text{ type}} [\![A[X := \Phi]]\!]$$

$$[\![\exists X.A]\!] = \bigcup_{\Phi \text{ type}} [\![A[X := \Phi]]\!]$$

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We consider a new *membership type* $t \in A$ (with t a term, A a type).

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- and allows the introduction of dependency.

The dependent function type $\forall x \in A.B$

- is defined as $\forall x.(x \in A \Rightarrow B)$,
- this is a form of *relativised quantification* scheme.

We also consider a new *restriction type* $A \upharpoonright P$:

- it is build using a type A and a “semantic predicate” P,
- $[\![A \upharpoonright P]\!]$ is equal to $[\![A]\!]$ if P is satisfied and to $[\![\bot]\!]$ otherwise.
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Semantic Restriction Type and Equalities

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Remark: refinement types $\{x \in A \mid P\}$ are encoded as $\exists x.(x \in A \upharpoonright P)$.

$$[\![A \Rightarrow B]\!] = \{\lambda x.w \mid \forall\, v \in [\![A]\!],\, w[x := v] \in [\![B]\!]\}$$

Interpretation of the Function Type

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Definition: we take $[\![A \Rightarrow B]\!] = \{\lambda x.t \mid \forall v \in [\![A]\!], t[x := v] \in [\![B]\!]^{\perp\!\!\!\perp}\}$.

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Pole and Orthogonality

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$$
\begin{aligned}
[\![A]\!] &\in \{\Phi \subseteq \Lambda_\iota \mid v \in \Phi \wedge v \equiv w \Rightarrow w \in \Phi\} \\
[\![A]\!]^{\perp\!\!\!\perp} &= \{\pi \in \Pi \mid \forall v \in [\![A]\!], v * \pi \in \perp\!\!\!\perp\} \\
[\![A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp} &= \{t \in \Lambda \mid \forall \pi \in [\![A]\!]^{\perp\!\!\!\perp}, t * \pi \in \perp\!\!\!\perp\}
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Value Restriction and Typing Judgments

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$$\frac{}{\Gamma, x : A; \Xi \vdash_{\text{val}} x : A} \qquad \frac{\Gamma, x : A; \Xi \vdash t : B}{\Gamma; \Xi \vdash_{\text{val}} \lambda x.t : A \Rightarrow B}$$

Theorem (adequacy lemma):

- if $\vdash t : A$ is derivable then $t \in [\![A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp}$,
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Proof by induction on the typing derivation.

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We only need to check that our typing rules are "correct".

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Proof by induction on the typing derivation.

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For example $\dfrac{\vdash_{val} v : A}{\vdash v : A}$ is correct since $[\![A]\!] \subseteq [\![A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp}$.

Adequacy of For All Introduction

$$\frac{\Gamma ; \Xi \vdash_{\text{val}} v : A}{\Gamma ; \Xi \vdash_{\text{val}} v : \forall X.A}\, X \notin \Gamma$$

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However we have $\cap_\Phi [\![A[X := \Phi]]\!]^{\perp\!\!\!\perp} \not\subseteq [\![\forall X.A]\!]^{\perp\!\!\!\perp} = \left(\cap_\Phi [\![A[X := \Phi]]\!]\right)^{\perp\!\!\!\perp}$.

Theorem (normalisation):

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Theorem (safety for simple datatypes):

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Theorem (safety for simple datatypes):

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Theorem (consistency):

there is no closed term $t : \bot$.

Part III

Semantical Value Restriction

Derived Rules for Dependent Functions

$$\frac{x : A \vdash t : B[a := x]}{\vdash_{\text{val}} \lambda x.t : \forall a \in A.B}$$

$$\frac{\vdash t : \forall a \in A.B \qquad \vdash_{\text{val}} v : A}{\vdash t\ v : B[a := v]}$$

Derived Rules for Dependent Functions

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$$\dfrac{\dfrac{\dfrac{\vdash t : \forall a{\in}A.B}{\vdash t : \forall a.(a{\in}A \Rightarrow B)}\,\text{Def}}{\vdash t : v{\in}A \Rightarrow B[a := v]}\,\forall_e \qquad \dfrac{\dfrac{\color{red}{\vdash_{\text{val}} v : A}}{\vdash_{\text{val}} v : v{\in}A}\,\in_i}{\vdash v : v{\in}A}\,\uparrow}{\vdash t\ v : B[a := v]}\,\Rightarrow_e$$

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Value restriction breaks the compositionality of dependent functions.

```
// add_n_Zero : ∀n∈nat, add n Zero ≡ n
add_n_Zero (add Zero S[Zero]) : add (add Zero S[Zero]) Zero ≡ add Zero S[Zero]
```

We replace $\dfrac{\vdash t : \forall a \in A.B \qquad \vdash_{\text{val}} v : A}{\vdash t\ v : B[a := v]}$ by $\dfrac{\vdash t : \forall a \in A.B \qquad \vdash u : A \qquad \vdash u \equiv v}{\vdash t\ u : B[a := u]}$.

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Can this rule be derived in the system?

We replace $\dfrac{\vdash t : \forall a \in A.B \qquad \vdash_{val} v : A}{\vdash t\ v : B[a := v]}$ by $\dfrac{\vdash t : \forall a \in A.B \qquad \vdash u : A \qquad \vdash u \equiv v}{\vdash t\ u : B[a := u]}$.

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The biorthogonal completion should not introduce new values.

Biorthogonal Completion Closed for Values

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The biorthogonal completion should not introduce new values.

The rule seems reasonable, but it is hard to justify semantically.

We do not have $v \in [\![A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp}$ implies $v \in [\![A]\!]$ in every realizability model.

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We extend the system with a new term constructor $\delta_{v,w}$ such that

$$\delta_{v,w} * \pi \succ v * \pi \qquad \text{iff} \qquad v \not\equiv w.$$

The New Instruction Trick

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- $w * [\lambda x.\delta_{x,v}]\varepsilon \succ \lambda x.\delta_{x,v} * w \,.\, \varepsilon \succ \delta_{w,v} * \varepsilon \succ w * \varepsilon \Downarrow$ if $w \in [\![A]\!]$

Problem: the definitions of $(\succ)$ and $(\equiv)$ are circular.

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We need to rely on a stratified construction of the two relations.

$$(\twoheadrightarrow_i) \quad = \quad (\succ) \cup \left\{(\delta_{v,w} * \pi, v * \pi) \mid \exists j < i, v \not\equiv_j w\right\}$$

$$(\equiv_i) \quad = \quad \left\{(t, u) \mid \forall j \leqslant i, \forall \pi, \forall \sigma, t\sigma * \pi \Downarrow_j \Leftrightarrow u\sigma * \pi \Uparrow_j\right\}$$

We then take

$$(\twoheadrightarrow) \quad = \quad \bigcup_{i \in \mathbb{N}} (\twoheadrightarrow_i) \qquad \text{and} \qquad (\equiv) \quad = \quad \bigcap_{i \in \mathbb{N}} (\equiv_i).$$

Part IV

Local Subtyping and Choice Operators

PML_2 is hard to implement for several reasons:

- it is a Curry-style language (quantifiers are not reflected in terms),
- many of its type constructors don't have "algorithmic contents".

A Syntax-directed Presentation

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Solution: handle these connectives using *local subtyping*.

We then obtain a type system with:

- one typing for each term (or value) constructor,
- one typing rule for each pair of type constructors (up to commutation).

We replace free variables with “choice operators”:

- $\varepsilon_{x \in A}(t \notin B)$ denotes some $v \in [\![A]\!]$ such that $[\![t[x := a]]\!] \notin [\![B]\!]^{\perp\!\!\!\perp}$ (if possible),
- and similar things are defined for types and other syntactic elements.
- Choice operators are interpreted using elements of the semantic domain.

Choice Operators and Local Subtyping

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We modify the system by:

- eliminating typing contexts (in favor of choice operators),
- introducing local subtyping judgments of the form $\Xi \vdash t : A \subseteq B$.
- They are interpreted as: "if $\Xi \vdash t : A$ holds, then $\Xi \vdash t : B$ also holds."

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Remark: $\Xi \vdash A \subseteq B$ can be encoded as $\Xi \vdash \varepsilon_{x \in A}(x \notin B) : A \subseteq B$.

$$\frac{\Xi \vdash \lambda x.t : A \Rightarrow B \subseteq C \qquad \Xi, \varepsilon_{x\in A}(t \notin B) \neq \square \vdash t[x := \varepsilon_{x\in A}(t \notin B)] : B}{\Xi \vdash \lambda x.t : C}{\Rightarrow_i}$$

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Examples of Syntax-directed Typing Rules

$$\frac{\Xi \vdash \lambda x.t : A \Rightarrow B \subseteq C \qquad \Xi, \varepsilon_{x\in A}(t \notin B) \neq \square \vdash t[x := \varepsilon_{x\in A}(t \notin B)] : B}{\Xi \vdash \lambda x.t : C}\Rightarrow_i$$

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$$\frac{\Xi \vdash t : A \Rightarrow B \qquad \Xi \vdash u : A}{\Xi \vdash t\ u : B} \Rightarrow_e$$

$$\frac{\Xi \vdash v : A \qquad \Xi \vdash C_k[v] : [C_k : A] \subseteq B}{\Xi \vdash C_k[v] : B} +_i \qquad \frac{\Xi \vdash v : \{l_k : A ; \dots\}}{\Xi \vdash v.l_k : A} \times_e$$

Examples of Syntax-directed (Local) Subtyping Rules

$$\frac{\Xi \vdash t : A[X := C] \subseteq B}{\Xi \vdash t : \forall X.A \subseteq B}\,\forall_l \qquad \frac{\Xi \vdash t : A \subseteq B[X := \varepsilon_X(t \notin B)] \qquad \Xi \vdash v \equiv t}{\Xi \vdash t : A \subseteq \forall X.B}\,\forall_r$$

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$$\frac{\Xi, u_1 \equiv u_2 \vdash t : A \subseteq B \quad \Xi \vdash v \equiv t}{\Xi \vdash t : A \restriction u_1 \equiv u_2 \subseteq B}\,\restriction_l \qquad \frac{\Xi \vdash t : A \subseteq B \quad \Xi \vdash u_1 \equiv u_2}{\Xi \vdash t : A \subseteq B \restriction u_1 \equiv u_2}\,\restriction_r$$

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$$\frac{\Xi, t \equiv u \vdash t : A \subseteq B \qquad \Xi \vdash t \equiv v}{\Xi \vdash t : u \in A \subseteq B}\,\in_l \qquad \frac{\Xi \vdash t : A \subseteq B \qquad \Xi \vdash t \equiv u \qquad \Xi \vdash t \equiv v}{\Xi \vdash t : A \subseteq u \in B}\,\in_r$$

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$$\frac{\Xi, w \neq \square \vdash w : A_2 \subseteq A_1 \qquad \Xi, w \neq \square \vdash t\ w : B_1 \subseteq B_2 \qquad \Xi \vdash t \equiv v}{\Xi \vdash t : A_1 \Rightarrow B_1 \subseteq A_2 \Rightarrow B_2}\,\Rightarrow$$

(where $w = \varepsilon_{x \in A_2}(t\ x \notin B_2)$)

Part V

Cyclic Proofs and Termination Checking

Recursive programs rely on a term $\varphi a.v$ (binding a term in a value).

$$\varphi a.v * \pi \quad \twoheadrightarrow \quad v[a := \varphi a\, v] * \pi \qquad\qquad \frac{\Xi \vdash v[a := \varphi a.v] : A}{\Xi \vdash \varphi a.v : A}\,\varphi$$

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Problem: we need to work with infinite proofs.

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Problem: we need to work with infinite proofs.

We introduce a cyclic structure in our proofs.

$$\frac{\forall \alpha\ (\Xi \vdash t : A)}{(\Xi \vdash t : A)[\alpha := \kappa]}\,\text{Gen} \qquad \frac{\dfrac{\dfrac{}{[\forall \alpha\ (\Xi \vdash t : A)]^{i}}}{\vdots}}{\dfrac{(\Xi \vdash t : A)[\alpha := \varepsilon_{\alpha}(t \notin A)]}{\forall \alpha\ (\Xi \vdash t : A)}\,\text{Ind}[i]}$$

$$\frac{\Xi \vdash t : A \subseteq B[X := \mu_{\infty}X.B]}{\Xi \vdash t : A \subseteq \mu_{\infty}X.B}\mu_{r,\infty}$$

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$$\frac{\Xi \vdash t : A \subseteq B[X := \mu_{\upsilon} X.B] \qquad \Xi \vdash \upsilon < \tau}{\Xi \vdash t : A \subseteq \mu_{\tau} X.B}\,\mu_{r}$$

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$$\frac{\Xi ; \tau > 0 \vdash t : A[X := \mu_{\varepsilon_{\theta<\tau}(t \in A[X := \mu_{\theta} X.A])} X.A] \subseteq B \qquad \Xi \vdash v \equiv t}{\gamma ; \Xi \vdash t : \mu_{\tau} X.A \subseteq B}\,\mu_{l}$$

Let us consider the "map" function: $\varphi m.\lambda f.\lambda l.[l \,|\, [\,] \rightarrow [\,] \,|\, x :: l \rightarrow f\ x :: m\ f\ l]$.

It can be given either of the types:

- $\forall X.Y(X \Rightarrow Y) \Rightarrow \mathrm{List}(X) \Rightarrow \mathrm{List}(X)$,
- $\forall \alpha.\forall X.Y(X \Rightarrow Y) \Rightarrow \mathrm{List}(\alpha, X) \Rightarrow \mathrm{List}(X)$,
- $\forall \alpha.\forall X.Y(X \Rightarrow Y) \Rightarrow \mathrm{List}(\alpha, X) \Rightarrow \mathrm{List}(\alpha, X)$.

$\mathrm{List}(\alpha, X)$ is defined as $\mu_\alpha L.[([\,]) : \{\} \,|\, (::) : X \times L]$.

Conclusion

Future Work

1) Practical issues (work in progress):
   - Composing programs that are proved terminating.
   - Extensible records and variant types (inference).

2) Toward a practical language:
   - Compiler using type information for optimisations.
   - Built-in types (`int64`, `float`) with their formal specification.

3) Theoretical questions:
   - Can we handle more side-effects? (mutable cells, arrays)
   - What can we realise with (variations of) $\delta_{v,w}$?
   - Can we extend the system with quotient types?
   - Can we formalise mathematics in the system?

Practical Subtyping for Curry-Style Languages

https://lepigre.fr/files/publications/LepRaf2018a.pdf

PML_2: Integrated Program Verification in ML

https://lepigre.fr/files/publications/Lepigre2018.pdf

Semantics and Implementation of an Extension of ML for Proving Programs

https://lepigre.fr/files/publications/Lepigre2017PhD.pdf

A Classical Realizability Model for a Semantical Value Restriction

https://lepigre.fr/files/publications/Lepigre2016.pdf

Thanks!